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HW 4

Math 575B
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1 Part II: Theory.

1. The linear system $y' = Ay$ with $y(0) = y_0$, where A is a symmetric matrix, is solved using Euler's Method.

   a.) Letting $e_n = y_n - y(nh)$ with $n = 0, 1, ...$ prove that

$$||e_n||_2 \leq ||y_0||_2 \max_{\lambda \in \sigma(A)} |(1 + h\lambda)^n - e^{nh\lambda}| \tag{1}$$

   where $\sigma(A)$ is the set of eigenvalues and $|| \cdot ||_2$ is the Euclidean or L^2 norm.

   Solution. First note that A *symmetric* $\implies A = D\Lambda D^{-1}$, where Λ is the diagonal matrix whose entries are the eigenvalues of A, for some unitary D.

   The exact solution $\vec{y}(nh) = \vec{y}(0)e^{Anh}$ which is approximately equivalent to $\vec{y}(nh) = \vec{y}(0)De^{\Lambda nh}D^{-1}$ for sufficiently small h

   The first Euler step will be:

$$\begin{aligned} \vec{y}_1 &= (I + hA)\vec{y}_0 \tag{2}\\ &= (I + hD\Lambda D^{-1})\vec{y}_0 \tag{3}\\ &= D(I + h\Lambda)D^{-1}\vec{y}_0 \tag{4} \end{aligned}$$

   Therefore the n^{th} step will be:

$$\begin{aligned} \vec{y}_n &= (D(I + h\Lambda)D^{-1})^n\vec{y}_0 \tag{5}\\ &= D^n(I + h\Lambda)^nD^{-n}\vec{y}_0 \tag{6} \end{aligned}$$

   Now look at $||\vec{e}_n||_2 = ||\vec{y}_n - \vec{y}(nh)||_2$

$$\begin{aligned} ||\vec{e}_n||_2 &\leq ||\vec{y}_0||_2||D^n(I + h\Lambda)^nD^{-n} - D^ne^{h\Lambda n}D^{-n}||_2 \tag{7}\\ &\leq ||\vec{y}_0||_2||D^n||_2||(I + h\Lambda)^n - e^{h\Lambda n}||_2||D^{-n}||_2 \tag{8}\\ \text{But D is unitary, so its norm is 1} & \tag{9}\\ &= ||\vec{y}_0||_2||(I + h\Lambda)^n - e^{h\Lambda n}||_2 \tag{10}\\ \text{Apply Hölders inequality} & \tag{11}\\ &\leq ||\vec{y}_0||_2||1||_1||(I + h\Lambda)^n - e^{h\Lambda n}||_\infty \tag{12}\\ &= ||\vec{y}_0||_2 \sup_{i \leq rank(A)} |(1 + h\lambda_i)^n - e^{nh\lambda_i}| \tag{13}\\ &= ||\vec{y}_0||_2|(1 + h\lambda_{max})^n - e^{nh\lambda_{max}}| \tag{14} \end{aligned}$$

b.) Demonstrate that for every $-1 << x <= 0$ and $n = 0, 1, \ldots$ it is true that

$$e^{nx} - \frac{1}{2}nx^2e^{(n-1)x} \leq (1+x)^n \leq e^{nx} \tag{15}$$

Solution. (Induction)
For $n = 0,\ 1 \leq 1 \leq 1$. So, it works at the first point. Assume the k^{th} point works
LHS:

$$e^{kx} - \frac{1}{2}kx^2e^{(k-1)x} \quad \leq \quad (1+x)^k \tag{16}$$

But, for allowed values of x

$$\begin{aligned} e^{kx} - \frac{1}{2}kx^2e^{(k-1)x} &\geq e^x(e^{kx} - \frac{1}{2}kx^2e^{(k-1)x}) & (17) \\ &\geq e^{(k+1)x} - \frac{1}{2}(k+1)x^2e^{kx} & (18) \end{aligned}$$

For $(1+x)^{k+1}$ term

$$\begin{aligned} e^{(k+1)x} - \frac{1}{2}(k+1)x^2e^{kx} &\leq (1+x)^ke^x - \frac{1}{2}x^2e^{kx} & (19) \\ &\leq (1+x)^ke^x & (20) \end{aligned}$$

Use Taylor expansion of e^x about 0 and it is easy to see that $e^x \leq 1 + x$ for allowed x. Therefore, the LHS reduces to

$$\begin{aligned} e^{(k+1)x} - \frac{1}{2}(k+1)x^2e^{kx} &\leq (1+x)^ke^x - \frac{1}{2}x^2e^{kx} & (21) \\ &\leq (1+x)^ke^x & (22) \\ &\leq (1+x)^{k+1} & (23) \end{aligned}$$

RHS:

$$\begin{aligned} (1+x)^k &\leq e^{kx} & (24) \\ \Longrightarrow (1+x)^ke^x &\leq e^{(k+1)x} & (25) \\ \Longrightarrow (1+x)^{k+1} &\leq e^{(k+1)x} & (26) \end{aligned}$$

By the same argument as for the LHS. Therefore, the induction argument is complete.

c.) Suppose that the maximal eigenvalue of A is $\lambda_{max} < 0$. Prove that, as $h \to 0$ and $nh \to t \in [0, t^*]$,

$$||e_n||_2 \leq \frac{1}{2}t\lambda_{max}^2e^{\lambda_{max}t}||y_0||_2h \leq \frac{1}{2}t^*\lambda_{max}^2||y_0||_2h \tag{27}$$

Solution.

2. Consider the linear stability of

$$y'' + ay' + (1 + b\cos(2\pi x))y = g(x) \qquad x \geq 0 \tag{28}$$

a.) Convert to first order form

Solution.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -(1 + b\cos(2\pi x)) & -a \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \tag{29}$$

b.) Assume $a > 0, |b| < 1$, and compute the eigenvalues for $g(x) = 0$

Solution. Consider the characteristic equation $\lambda^2 - tr(A)\lambda + det(A)$ where A is the 2×2 matrix from part a.)
Setting g(x) = 0 and solving for λ yields

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4(1 + b\cos(2\pi x)}}{2} \tag{30}$$

c.) Compare the results of stability theory given from the homogeneous equation to those obtained from the linear constant coefficient model equation $y' = \lambda y + g(x)$ as applied to the above system.

Solution. Given the constraints on a and b, the real part of the eigenvalues are always negative (a decaying oscillation), i.e. the solution is stable.

2 Part III: Experiment.

1. Solve IVP with Adams-Moulton 4-step:

$$\begin{aligned} y' &= \cos(t) + \sin(3t) && (31) \\ y(0) &= 1, \qquad 0 \leq t \leq 1 && (32) \\ & && (33) \end{aligned}$$

Solution. There is a typo. The differential equation should be

$$\begin{aligned} y' &= \cos(2t) + \sin(3t) && (34) \\ y(0) &= 1, \qquad 0 \leq t \leq 1 && (35) \\ & && (36) \end{aligned}$$

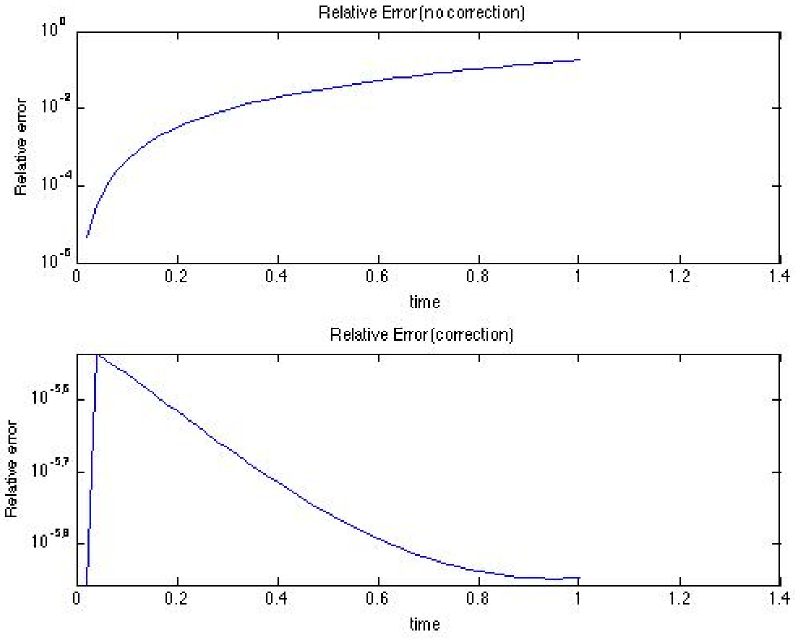

Figure 1: Rel. Err. vs. x

It is clear that the exact solution to the problem used is inaccurate for the top plot. The relative error should converge to something constant. The relative error in the top plot keeps growing while the second plot shows convergence to a constant.

2. Compute an approximation to

$$\frac{\partial x_1}{\partial t} = \sin(x_1) + \cos(tx_2) \tag{37}$$

$$\frac{\partial x_2}{\partial t} = t^{-1} \sin t(x_1) \tag{38}$$

$$x_1(-1) = 2.37, \tag{39}$$

$$x_2(-1) = -3.48 \qquad -1 \leq t \leq 1 \tag{40}$$

Solution. a.) The discontinuity is removable, therefore there should be no question of existence and uniqueness of solution. The partial derivatives are otherwise continuous and differentiable.
b.) My strategy to deal with the singularity is to use L'Hospital's rule in an ϵ ball around $t = 0$ This way the singularity can be dealt with as an approximation.

c.) From the look of it, the convergence rate is approximately between h^3 and h^4
d.) Code works. I have the flu right now and I feel like I am knocking on death's door..

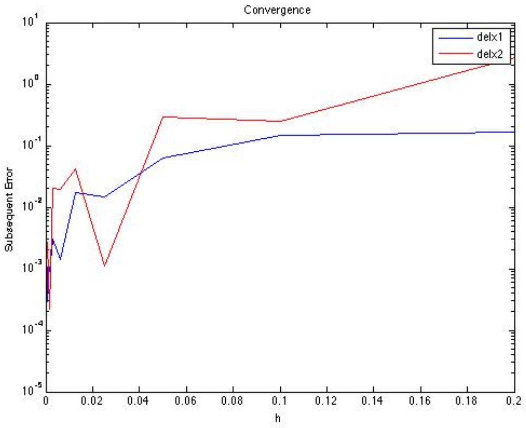

Figure 2: Error plot